Lecture 6

Lyapunov-based Control Design

6.1. Energy Functions

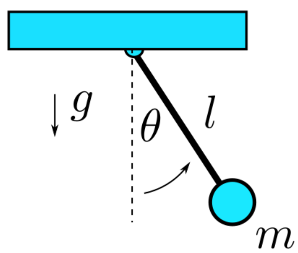

Figure 6.1: The simple pendulum.

Consider the simple one degree-of-freedom pendulum system, given by the dynamics

$$ml^2\ddot{\theta} + mgl\sin\theta = -b\dot{\theta}. \tag{6.1}$$

where b is the damping coefficient, m is the mass, and l is the link length. We can write the energy of this system as

$$E(\theta, \dot{\theta}) = \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta \tag{6.2}$$

The derivative of this energy is

$$\dot{E} = -b\dot{\theta}^2 \leq 0. \tag{6.3}$$

Note the that the derivative of the energy is negative except when $\dot{\theta} = 0$. What can we say about its stability?

6.2. Notions of Stability

How can we describe notions of stability and instability?

6.2.1 Local Stability of a Fixed Point

What does it mean to be stable in the the sense of Lyapunov (i.s.L)? x^* is stable *i.s.L.* if for any $\delta > 0$, $\exists \epsilon > 0$ such that

$$||x(0) - x^*|| < \delta \implies ||x(t) - x^*|| < \epsilon \quad , \forall t. \tag{6.4}$$

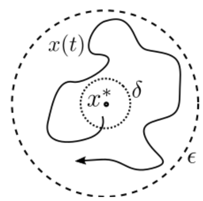

Figure 6.2: Stability in the sense of Lyapunov.

6.2.2 Asymptotic Stability

$\exists \epsilon > 0$ such that

$$||x(0) - x^*|| < \epsilon \implies x(t) \to x^* \text{ as } t \to \infty. \tag{6.5}$$

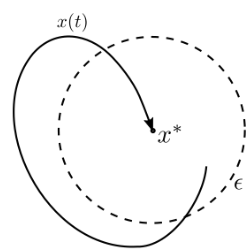

Figure 6.3: Asymptotic stability.

6.2.3 Exponential Stability

An equilibrium point $x*$ is exponential stable if there exists $\alpha > 0$ and $\lambda > 0$ such that

$$||x(t) - x^*|| \leq \alpha ||x(0) - x^*|| e^{-\lambda t}. \tag{6.6}$$

6.3. Local Lyapunov Stability

Consider the system

$$\dot{x} = f(x). \tag{6.7}$$

If f is continuous, and there exists $V(x)$ for some $\mathcal{D}$ around the origin such that

$$V(x) > 0, \quad \forall x \in \mathcal{D} \tag{6.8}$$

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) \leq 0, \quad \forall x \in \mathcal{D} \tag{6.9}$$

then the origin is stable in the sense of Lyapunov (i.s.L.).
If

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) < 0 \quad \forall x \in D, \tag{6.10}$$

then the origin is locally asymptotically stable.
If

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) \leq -\alpha V(x) \quad \forall x \in D, \quad \alpha > 0 \tag{6.11}$$

then the origin is locally exponentially stable.

6.4. Global Lyapunov Stability

If

$$V(x) > 0, \tag{6.12}$$

$$V(x) = \frac{\partial V}{\partial x} f(x) < 0, and \tag{6.13}$$

$$V(x) \longrightarrow \infty \quad \text{whenever} \quad ||x|| \longrightarrow \infty \tag{6.14}$$

over the whole space, the origin $x = 0$ is globally asymptotically stable.
If

$$\dot{V}(x) < -\alpha V(x) \tag{6.15}$$

$$\tag{6.16}$$

where $\alpha > 0$ the system is globally exponentially stable.
The new condition refers to $V(x)$ being radially unbounded, which means that trajectories can not diverge as $V(x)$ decreases.

6.5. LaSalle's Theorem

Given a system $\dot{x} = f(x)$ with f continuous. If we can produce a scalar function $V(x)$ with continuous derivatives for which we have

$$V(x) > 0, \quad \dot{V}(x) \leq 0, \tag{6.17}$$

and $V(x) \longrightarrow \infty$ as $||x|| \longrightarrow \infty$, then x will converge to the largest invariant set where $\dot{V}(x) = 0$.

6.6. Lyapunov Functions and the Cost-to-Go

$$0 = \min_u \left[g(x, u) + \frac{\partial J^*}{\partial x} f(x, u) \right] \tag{6.18}$$

If we can optimize for $u^*(x)$, then we are left with $0 = g(x, u^*) + \frac{\partial J^*}{\partial x} f(x, u^*)$ or

$$\dot{J}^*(x) = -g(x, u^*). \tag{6.19}$$

If $g(x, u^*)$ is positive definite and zero only at $x = 0$, the cost-to-go is a Lyapunov function that guarantees asymptotic stablity.

Optimal Control	Lyapunov-based Control
$\dot{J}^*(x) = -g(x, u^*)$	$\dot{V}(x) < 0$

Figure 6.4: Cost-to-go vs. Lyapunov function

6.7. Lyapunov Analsis for Linear Systems

If you have a linear system

$$\dot{x} = Ax \tag{6.20}$$

that satisfies

$$V(x) = x^T S x, \quad S = S^T > 0, \tag{6.21}$$

and

$$\dot{V}(x) = x^T SAx + x^T A^T Sx < 0 \tag{6.22}$$

then the origin is Globally Asymptotically Stable (G.A.S)

6.8. Lyapunov Analysis via Sums-of-Squares Programming

How can we check if a polynomial is non-negative everywhere? This in general is NP-Hard (i.e., not solvable or verifiable in polynomial time). However, there is a sufficient condition to prove non-negativity. If I can write a polynomial as a sum-of-squares, then I can prove it is non-negative.

Consider the polynomial

$$p(x) = 2 - 4x + 5x^2. \tag{6.23}$$

I can write this polynomial as

$$p(x) = 1 + (1 - 2x)^2 + x^2. \tag{6.24}$$

This can also be written as

$$p(x) = \begin{bmatrix} 1 \\ x \end{bmatrix}^T \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix}. \tag{6.25}$$

If one can find a positive semi-definite matrix S such that $p(x) = m(x)^T S(x) m(x)$ then $p(x)$ is non-negative.

The search for this positive semi-definite matrix can be implemented as a semi-definite program and solved for efficiently using interior point methods.

A semi-definite program has the structure where the objective is a linear function subject to a matrix inequality. It is given as

$$\min \quad c^T x \tag{6.26}$$

$$s.t. \quad F(x) \geq 0 \tag{6.27}$$

$$F(x) = F_0 + \sum_{i=1}^{N} x_i F_i. \tag{6.28}$$

We can search for a Lyapunov function using semi-definite programming!

$$V_\alpha(x) = \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_1 x_2 ... \tag{6.29}$$
$$s.t. \quad V_\alpha \text{ is SOS}, \quad V_\alpha(0) = 0 \tag{6.30}$$
$$-\dot{V}_\alpha(x) = -\frac{\partial V_\alpha}{\partial x} f(x) \text{ is SOS} \tag{6.31}$$

The constraint $\dot{V}_\alpha(0)$ is already implicitly satisfied given the constraints on V_α if V_α is a smooth polynomial.

This only proves stability of the origin. If we want to show asymptotic stability we must satisfy

$$-\dot{V}_\alpha(x) - \epsilon x^T x \text{ is SOS.} \tag{6.32}$$

where $\epsilon > 0$.

6.9. Local Lyapunov Functions

It is challenging to find global controllers for real-world control problems.

Instead, let's focus on a region

$$B = \left[x \middle| V(x) \leq \rho\right] \implies \dot{V} = \frac{\partial V(x)}{\partial x} f(x) \leq 0, \tag{6.33}$$

where $V(x)$ is positive definite. To enforce this constraint, we apply the S-procedure.

Generally stated, the S-procedure is a means of enforcing the relationship

$$f(x) \leq 0 \implies g(x) \leq 0. \tag{6.34}$$

For any two functions $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$, if there exists $\lambda \geq 0$ such that $g(x) \leq \lambda f(x)$ for all x, then for any x satisfying $f(x) \leq 0$, we must have $g(x) \leq 0$.
Proof: If $f(x) \leq 0$, then $\lambda f(x) \leq 0$. Since $g(x) \leq \lambda f(x)$, $g(x) \leq 0$ whenever $f(x) \leq 0$.

We desire

$$V(x) \leq \rho \implies \dot{V} \leq 0 \tag{6.35}$$

in some region B. Using the S-procedure, we can write

$$\dot{V} \leq \lambda(V(x) - \rho) \tag{6.36}$$
$$\dot{V} - \lambda(V(x) - \rho) \leq 0, \tag{6.37}$$

where $\lambda > 0$. This can then be enforced using sum-of-squares programming

$$-\dot{V} + \lambda(V - \rho) \quad \text{is SOS}, \quad \lambda \text{ is SOS}, \quad V > 0. \tag{6.38}$$

The positive definite contraint on V could be enforced by the following constraint

$$V - \gamma x^T x \text{ is SOS}, \quad \gamma > 0. \tag{6.39}$$

6.10. Bilinear Alternations

In examining Eq. 6.38, one may notice that the first SOS constraint is non-covex in the decision parameters (i.e., the parameters in V, λ, and ρ).

We would like to execute the optimization program

$$\begin{aligned}
\max_{\rho,V,\mu} \quad & \rho \\
\text{s.t.} \quad & \rho > 0, \\
& -\dot{V} + \mu(V-\rho) \in \Sigma \\
& \mu \in \Sigma \\
& V > 0
\end{aligned}$$

where Σ denotes the set of sum-of-squares polynomials, and $V(0) = 0$ by construction.
However, due to the non-convexity of the SOS constraint, we must adopt a bilinear alternation strategy.

In the first step, we hold paramters of V and ρ fixed and optimize the multiplier. We maximize γ, which serves as a positive slack variable.

$$\begin{aligned}
\max_{\mu,\gamma} \quad & \gamma \\
\text{s.t.} \quad & \gamma > 0, \\
& -\dot{V} + \mu(V-\rho) - \gamma \in \Sigma
\end{aligned}$$

In the second step, we hold parameters of the multiplier μ fixed and optimize the the parameters of the Lyapunov function ρ and V fixed and attempt to maximize the volume of the invariant region through a surrogate like ρ.

$$\begin{aligned}
\max_{\rho,V} \quad & \rho \\
\text{s.t.} \quad & \rho > 0, \\
& -\dot{V} + \mu(V-\rho) \in \Sigma \\
& V > 0.
\end{aligned}$$

How would we come up with the initial set of parameters for V and ρ?

6.11. Control Lyapunov Functions

Control Lyapunov Functions are an extension of Lyapunov functions that explicitly consider control inputs. Generally, for a nonlinear system $\dot{x} = f(x,u)$, they guarantee that

$$\forall x \neq 0, \quad \exists u \tag{6.40}$$

$$\text{s.t. } \dot{V}(x,u) = \frac{\partial V}{\partial x} f(x,u) < 0. \tag{6.41}$$

$$\tag{6.42}$$

and

$$\exists u, \quad \dot{V}(0,u) = 0. \tag{6.43}$$

Consider a special class of nonlinear dynamical systems:

$$\dot{x} = f(x) + g(x)u. \tag{6.44}$$

This dynamical system is said to be "control-affine".
A poitive definite function $V(x)$ is a Control Lyapunov Function for the system in 6.44 if

$$\frac{\partial V}{\partial x} g(x) = 0 \text{ for } x \in D, x \neq 0 \implies \frac{\partial V}{\partial x} f(x) < 0. \tag{6.45}$$

For this system and Control Lyapunov Function, a control policy, $u = \pi(x)$, can also be given as

$$\pi(x) = \begin{cases} -\frac{\frac{\partial V}{\partial x} f + \sqrt{(\frac{\partial V}{\partial x} f)^2 + (\frac{\partial V}{\partial x} g)^4}}{\frac{\partial V}{\partial x} g}, & \frac{\partial V}{\partial x} g \neq 0 \\ 0, & \frac{\partial V}{\partial x} g = 0 \end{cases} \tag{6.46}$$

Bibliography

[1] Russ Tedrake. *Underactuated Robotics*. 2023.