

## Lecture 7

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# Contraction-based Control Design

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Sometimes, when the goal is trajectory tracking and not regulation of a fixed-point, contraction analysis (instead of Lyapunov analysis) provides a better framework for understanding controller performance.

### 7.1. Differential Notion of Stability

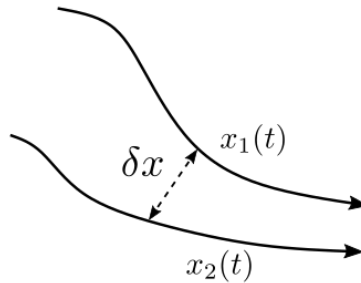


Figure 7.1: Virtual displacement between two trajectories.

Given  $\dot{x} = f(x)$ , let us denote the infinitesimal displacement between the two trajectories as  $\delta x$ , which is known as a virtual displacement. If  $f(x)$  is continuously differentiable then

$$\delta \dot{x} = \frac{\partial f}{\partial x} \delta x. \quad (7.1)$$

We would like to be able to show that this virtual displacement tends to zero with time. Consider the metric  $(x^T \delta x)$ . We can show that

$$\frac{d}{dt}(\delta x^T \delta x) = 2\delta x^T \delta \dot{x} \quad (7.2)$$

$$= 2\delta x^T \frac{\partial f}{\partial x} \delta x. \quad (7.3)$$

If  $\frac{\partial f}{\partial x}$  is uniformly negative definite (u.n.d.) in a region,  $\delta x^T \delta x$  converges to zero exponentially at rate  $\lambda$ , i.e., the system is contracting in that region. What does it mean to be uniformly negative definite?

$$\exists \lambda > 0, \forall x, \forall t \geq 0, \frac{1}{2} \left( \frac{\partial f}{\partial x} + \frac{\partial f^T}{\partial x} \right) \leq -\lambda I < 0 \quad (7.4)$$

Here  $\frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{\partial f^T}{\partial x}\right)$  is the symmetric part of  $\frac{\partial f}{\partial x}$ . We can now write

$$\frac{d}{dt}(\delta x^T \delta x) \leq -2\lambda \delta x^T \delta x. \quad (7.5)$$

If a system is *strictly* contracting, we have

$$\frac{d}{dt}(\delta x^T \delta x) < -2\delta x^T \lambda \delta x. \quad (7.6)$$

Any square matrix  $M$  can be decomposed into a symmetric and skew symmetric part where

$$M = \frac{1}{2}(M + M^T) + \frac{1}{2}(M - M^T). \quad (7.7)$$

The quadratic function associated with a skew symmetric matrix  $M$  where  $M = -M^T$  is zero, since  $x^T M x = -x^T M^T x$ , which is only satisfied by  $x^T M x = 0$ .

However, it may be challenging to guarantee, even for stable systems that  $\frac{\partial f}{\partial x}$  is u.n.d.

**Theorem:** Given the system  $\dot{x} = f(x)$ , any trajectory which starts in a ball of constant radius centered about a given trajectory and contained at all times in a contraction region, remains in that ball and converges exponentially to this trajectory.

**Proof:** Consider the error dynamics  $\tilde{x} = x_1 - x_2$ . Then

$$\dot{\tilde{x}} = f(\tilde{x}) = \int_0^1 \frac{\partial f}{\partial x}(x_1 + \lambda \tilde{x}) d\lambda \tilde{x}. \quad (7.8)$$

Also consider the Lyapunov function  $V = \tilde{x}^T \tilde{x}$ .

$$\dot{V} = \frac{d}{dt}(\tilde{x}^T \tilde{x}) = 2\tilde{x}^T \int_0^1 \frac{\partial f}{\partial x}(x_1 + \lambda \tilde{x}) d\lambda \tilde{x} \leq -2\lambda \tilde{x}^T \tilde{x}. \quad (7.9)$$

## 7.2. Contraction Metrics

Let's generalize the notion of contraction. Consider the contraction metric

$$M : \mathbb{R}^n \rightarrow S_n > 0 \quad (7.10)$$

where  $S_n$  is a positive definite matrix.

We can then define

$$\delta x^T M(x) \delta x, \quad (7.11)$$

which is a Riemannian squared length (i.e., a smoothly varying inner product).

Given this metric, what is the condition to guarantee contraction?

Consider a change of coordinates:

$$\delta z = \Theta \delta x. \quad (7.12)$$

Taking the time derivative gives

$$\frac{d}{dt} \delta z = \dot{\Theta} \delta x + \Theta \delta \dot{x} \quad (7.13)$$

$$= (\dot{\Theta} + \Theta \frac{\partial f}{\partial x}) \Theta^{-1} \delta z \quad (7.14)$$

$$= F \delta z. \quad (7.15)$$

The generalized Jacobian is

$$F = (\dot{\Theta} + \Theta \frac{\partial f}{\partial x}) \Theta^{-1}. \quad (7.16)$$

Given the metric

$$\delta z^T \delta z = \delta x^T M \delta x, \quad (7.17)$$

where  $M = \Theta^T \Theta$ , we have

$$\frac{d}{dt} \delta z^T \delta z = 2 \delta z^T F \delta z \leq -\lambda \delta z^T \delta z, \quad \lambda > 0. \quad (7.18)$$

If  $F$  is uniformly negative definite, we can similarly conclude the exponential convergence of two trajectories.

We can also write

$$\frac{d}{dt} \delta z^T \delta z = \frac{d}{dt} (\delta x^T M \delta x) = \delta x^T \left( \frac{\partial f^T}{\partial x} M + \dot{M} + M \frac{\partial f}{\partial x} \right) \delta x \leq -\lambda \delta z^T \delta z \quad (7.19)$$

$$= \delta x^T \left( A^T M + \dot{M} + M A \right) \delta x \leq -\lambda \delta x^T M \delta x, \quad (7.20)$$

where  $A = \frac{\partial f}{\partial x}$ .

If  $A^T M + \dot{M} + M A \leq -\lambda M$ ,  $\lambda > 0$ , also guarantees convergence of trajectories in a region.

Riemannian geometry extends Euclidean distances/angles to smooth nonlinear manifolds.

Contraction is useful for ‘‘analysis,’’ but what about synthesis?

### 7.3. Control Contraction Metrics

Let’s return to the control-affine dynamical systems considered during the discussion of control Lyapunov functions, which are given as

$$\dot{x} = f(x) + B(x)u. \quad (7.21)$$

This system has differential dynamics given by

$$\dot{\delta x} = A(x, u) \delta x(t) + B(x) \delta u(t). \quad (7.22)$$

where

$$A(x, u) = \frac{\partial}{\partial x} (f(x) + B(x)u). \quad (7.23)$$

If there exists a bounded symmetric positive matrix function  $M(x)$  and

$$\delta x^T M B = 0 \implies \delta x^T (\dot{M} + A^T M + M A + 2\lambda M) \delta x < 0, \quad (7.24)$$

$\forall x, \forall t$ , the system is universally exponentially stabilizable with rate  $\lambda > 0$ . Note that the control contraction metric theorem specifies that the system is strictly contracting.

Consider the time derivative

$$\frac{d}{dt} (\delta x^T M \delta x) = \delta x^T \dot{M} x + \delta x^T M \delta \dot{x} + \delta \dot{x}^T M \delta x \quad (7.25)$$

$$= \delta x^T \dot{M} x + \delta x^T M (A \delta x + B \delta u) + (A \delta x + B \delta u)^T M \delta x \quad (7.26)$$

$$= \delta x^T (\dot{M} + A^T M + M A) \delta x + \delta x^T M B \delta u + \delta u^T B^T M \delta x \quad (7.27)$$

$$= \delta x^T (\dot{M} + A^T M + M A) \delta x + 2 \delta x^T M B \delta u. \quad (7.28)$$

When taking the time derivative of a quadratic form like  $V = x^T P x$ , where  $\dot{x} = A x$  and  $P$  is symmetric positive definite, we can write

$$\frac{d}{dt} V = \dot{x}^T P x + x^T P \dot{x} \quad (7.29)$$

$$= x^T A^T P x + x^T P A x. \quad (7.30)$$

Since this is a quadratic form, we also have

$$x^T A^T P x + x^T P A x = 2x^T P A x. \quad (7.31)$$

However

$$A^T P + P A \neq 2P A. \quad (7.32)$$

This is because  $2P A$  may have a skew-symmetric part that is only zero when used in a quadratic form.

If  $\delta x^T M B \neq 0$ , the system can be made negative definite by  $\delta u$ .

If  $\delta x^T M B = 0$ , the system is naturally contracting.

A “differential feedback”  $\delta k(x, \delta x, u, t)$  exists that achieves exponential stabilization of the dynamics.

A control signal can be constructed by integrating  $\delta u = \delta k(x, \delta x, u, t)$  along the shortest path according to the metric (i.e., the geodesic) as follows:

$$u(t, s) = u^*(t) + \int_0^s \delta k(\gamma(s), \frac{\partial \gamma}{\partial s}(s), u(t, s)) ds. \quad (7.33)$$

Here  $\gamma(s)$  is the geodesic,  $\frac{\partial \gamma(s)}{\partial s}$  is the gradient of the geodesic along the path. If  $\gamma$  is a geodesic, then the path minimizes the Riemannian energy

$$E(\gamma) = \int_0^1 \left( \frac{\partial \gamma}{\partial s} \Big|_s^T M(\gamma(s)) \frac{\partial \gamma}{\partial s} \Big|_s \right) ds. \quad (7.34)$$

$E(\gamma)$  to the target state is also a control Lyapunov function.

#### 7.4. Dual Control Contraction Metric and Convex Optimization

To be able to formulate a convex optimization problem, we must use a dual metric  $W(x)$ , where

$$\eta = M(x) \delta x \quad \text{and} \quad W(x) = M(x)^{-1}. \quad (7.35)$$

Writing the CCM constraints in terms of  $\eta$  and  $W(x)$ , we have

$$\eta^T B = 0 \implies \eta^T (-\dot{W} + A W + W A^T + 2\lambda W) \eta < 0. \quad (7.36)$$

Pointwise linear matrix inequalities must hold for all  $x$  and can be approximated with SOS.

The above can be written in reduced form as

$$B_{\perp}^T (-\dot{W} + A W + W A^T + 2\lambda W) B_{\perp} < 0 \quad (7.37)$$

where

$$B_{\perp}^T B = 0. \quad (7.38)$$

Assuming a simpler differential control law of the form  $\delta u = K(x) \delta x$ , a dual CCM can be computed using slightly stronger conditions:

$$-\dot{W} + A W + W A^T - \rho B B^T + 2\lambda W \leq 0 \quad \text{CONDITION 1} \quad (7.39)$$

and

$$\sum_j \frac{\partial W}{\partial x_j} b_{i,j} - \frac{\partial b_i}{\partial x} W - W \frac{\partial b_i}{\partial x} = 0 \quad \forall i \quad \text{CONDITION 2} \quad (7.40)$$

where  $b_i$  is the  $i^{\text{th}}$  column of  $B$ . Under these conditions

$$K = -\frac{1}{2} \rho B^T W^{-1}. \quad (7.41)$$

## Bibliography

- [1] Ian R Manchester and Jean-Jacques E Slotine. Control contraction metrics: Convex and intrinsic criteria for nonlinear feedback design. *IEEE Transactions on Automatic Control*, 62(6):3046–3053, 2017.
- [2] Winfried Lohmiller and Jean-Jacques E Slotine. On contraction analysis for non-linear systems. *Automatica*, 34(6):683–696, 1998.